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Nonlinear Electrodynamics and NED-Inspired Chiral Solitons

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A nonlinear electrodynamics Lagrangian is considered and the electric field associated to an electric point-like charge is derived. The corresponding expression for the total field energy which has finite value in our model, is obtained. The chiral form of the Lagrangian is also presented. Topological, finite-energy, spherically symmetric solutions of the chiral model are studied and some of their properties are discussed.

KEY WORDS: nonlinear electrodynamics; chirality; chiral solitons.

1. INTRODUCTION

Nonlinear models have been of much interest in different branches of physics. As a well-known example in classical electrodynamics, one could mention the nonlinear model introduced by Born and Infeld (1934). Born-Infeld model has some interesting properties. In particular, a spherically symmetric field configuration has a finite energy, in contrast to the conventional (linear) electrodynamics. Born and Infeld's theory, contains some arguments about the necessity of a nonlinear generalization of electrodynamics and possible connection between such theories and quantum mechanics.

In recent years, nonlinear models are attracting more and more attention. Skyrme formulated a unified theory for baryons and mesons in terms of a chiral field (Skyrme, 1955, 1961). His theory analyzes different aspects of baryons and mesons and also their interactions. We know that in the large-N limit, QCD is equivalent to an effective meson theory (t'Hooft, 1974; Adkins *et al.*, 1983), like the Skyrme model where chiral solitons of such theories reproduce the static properties of real baryons, approximately. Chiral Born-Infeld solitons share this interesting property.

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Effective meson field theory can be reformulated in terms of the chiral field $U = \exp(i \frac{\phi_\pi . \tau}{f_\pi})$, where ϕ_π is associated with π-mesons, $f_\pi = 93$ MeV is the pion decay constant and τ_i's are the Pauli matrices. Skyrme modified the nonlinear sigma model by adding a non-minimal term to it, in order to prevent the solitons from shrinking to zero-size. He introduced the Lagrangian

$$L = \frac{1}{4} f_\pi^2 \mathrm{Tr}(\partial_\mu U \partial^\mu U^\dagger) + \frac{1}{32e^2} \mathrm{Tr}[(\partial_\mu U)U^\dagger, (\partial_\nu U)U^\dagger]^2, \tag{1}$$

where $L_\mu = U^\dagger \partial_\mu U$ is a Cartan left-invariant form (Zahed and Brown, 1986; Nikolaev, 1989). The last term which contains the dimensionless parameter e was introduced by Skyrme to stabilize the soliton solutions. This theory has a set of topological static solutions which can be classified by the value of the topological (baryon) charge

$$B = \frac{1}{24\pi^2} \int d^3 \epsilon_{ijk} \mathrm{Tr}(L_i L_j L_k). \tag{2}$$

The soliton solutions of this model are associated with baryon states with different baryon charges. For instance, the soliton with topological charge $B = 1$ is considered as the nucleon.

Another chiral soliton model for the description of baryons was proposed by O.V. Pavlovsky in Pavlovsky (2002). He considered a model with the Lagrangian

$$\mathcal{L}_{\mathrm{ChBI}} = -f_\pi^2 \mathrm{Tr} \beta^2 \left(1 - \sqrt{1 - \frac{1}{2\beta^2} L_\mu L^\mu}\right), \tag{3}$$

where β is the mass dimensional scale parameter of the model. The bag formation of this theory is worked out in Pavlovsky (2003).

It is desirable that such a theory have stable and finite-energy soliton solutions. The theory should be Lorentz- and chiral-invariant. The model (3) is motivated by the work of Born and Infeld. Pavlovsky has directly used the form of Born-Infeld action to set his model and derive the topological solitons.

In this article, we use, instead, a Lagrangian density quadratic in the field tensor. Whatsoever nonlinear function of F^2 is used in the Lagrangian, one expects (from a Taylor expansion of this function) to have a next-important term of second order in F^2. Starting from such a Lagrangian, we first obtain spherically symmetric solutions in the case of pure radial electric field. We also use this form of the Lagrangian for an effective meson theory. We show that in the case of pure radial electric field, the total energy of the field has a finite value, in spite of the mild singularity in the E-field.

We will study the direct analogue of the nonlinear electrodynamics model for a chiral field. The resulting chiral model does not have singularities and has a set of stable topological solutions. We find the finite-energy soliton solutions of a spherically symmetrical configuration within this model, using numerical calculations.

2. NONLINEAR ELECTRODYNAMICS

In this section, we introduce a nonlinear electrodynamics which shows attractive properties despite its simplicity. The total energy associated with the localized charge is finite in free space. The proposed Lagrangian has the following form

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \lambda(F^{\mu\nu}F_{\mu\nu})^2. \tag{4}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ as in the Maxwell's electrodynamics and λ is the constant parameter of the model. Obviously, the action (4) reduces to the usual Maxwell form in the $\lambda \to 0$ limit. In the case of a pure radial electric field, we find electric field E as a function of spherical coordinate r by using the Euler-Lagrange equation

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0. \tag{5}$$

Substituting for $\mathcal{L}$ we obtain

$$\partial_\mu F^{\mu\nu} = -\frac{J^\nu}{c}, \tag{6}$$

where c is the speed of light in free space and

$$\frac{J^\nu}{c} = F^{\mu\nu}\partial_\mu \ln(8\lambda F^{\alpha\beta}F_{\alpha\beta} - 1). \tag{7}$$

It is seen that the electromagnetic current is generated by the EM-field itself, as expected in a NED. From now on, we use the Heaviside-Lorentz system of units.

Substituting $F_{i0} = E_i$ in (5) and considering the radial ansatz $E_r = E(r)$ we obtain

$$E^3(r) + \frac{E(r)}{16\lambda} - \frac{\mathcal{C}^2}{4r^2} = 0, \tag{8}$$

where $\mathcal{C}$ with the dimension $[x]^{-2}$ is the constant of integration. By solving the latter equation, we obtain

$$E(r) = \frac{\left[\left(72\,\mathcal{C}^2\,\lambda + \sqrt{\frac{3\,r^4+1728\,\mathcal{C}^4\,\lambda^3}{\lambda}}\right)\lambda^2 r\right]^{2/3} - r^2\sqrt[3]{3}\lambda}{12\lambda r\left[\left(72\,\mathcal{C}^2\,\lambda + \sqrt{\frac{3\,r^4+1728\,\mathcal{C}^4\,\lambda^3}{\lambda}}\right)\lambda^2 r\right]^{1/3}}\sqrt[3]{3}. \tag{9}$$

The field $E(r)$ diminishes as r increases only for $\lambda > 0$ since the coefficient of the leading term in the numerator is proportional to $(\lambda^2/\sqrt{|\lambda|})^{\frac{2}{3}} - \lambda$ which vanishes for positive values of λ. For simplicity, we investigate the asymptotic behavior of the electric field, $E(r)$, just for $\lambda = -1$. The electric field behavior near $r = 0$ is

$$E(r \to 0) = \left(\frac{\mathcal{C}}{2r}\right)^{\frac{2}{3}} - \frac{1}{48}\left(\frac{2r}{\mathcal{C}}\right)^{\frac{2}{3}} + \frac{1}{331776}\left(\frac{2r}{\mathcal{C}}\right)^{\frac{10}{3}} + \cdots \tag{10}$$

and for large values of r, the electric field $E(r)$ is proportional to $1/r^2$ as expected,

$$E(r \to \infty) = 4\left(\frac{\mathcal{C}}{r}\right)^{2} - 1024\left(\frac{\mathcal{C}}{r}\right)^{6} + 786432\left(\frac{\mathcal{C}}{r}\right)^{10} + \cdots. \tag{11}$$

Comparing the first term in (11) with the Maxwell's electric field due to a point charge, one can find the value of constant $\mathcal{C}$. Since we have used the Heaviside-Lorentz system, we have:

$$\mathcal{C}^2 = \frac{Q}{16\pi}. \tag{12}$$

It is an easy task to compute the T^{00} component of the energy momentum tensor $T^{\mu\nu}$

$$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\rho)}\partial^\nu A_\rho - g^{\mu\nu}\mathcal{L}, \tag{13}$$

where $g^{\mu\nu}$ is the metric tensor. Substituting for $\mathcal{L}$ we obtain

$$T^{00} = \frac{1}{2}E^2 - 4\lambda E^4 = T^{00}_{\text{Maxwell}} - 4\lambda E^4, \tag{14}$$

which also reduces to the Maxwell form in the limit $\lambda \longrightarrow 0$. It is obvious from the asymptotic behaviors (10), (11) of the electric field E that the integral of T^{00} over the whole space which is the total energy of the system, has a finite value and does not diverge. Since the Coulomb force between two point charges can be attributed to the $\mathbf{E}_1.\mathbf{E}_2$ term in the field energy density (Jackson, 1975), one expects that the force between two localized solutions of the NED model at large inter-charge distances should be inverse square. On small distances, however, deviations from the inverse-square force is expected.

3. CHIRAL FIELD SOLITONS

In the previous section, we studied the properties of the proposed NED model for the field of a localized electric charge in free space. Based on the results of the previous section, we are motivated to use that form of the Lagrangian to make a chiral field model in analogy with the work (Pavlovsky, 2002). The model should have finite energy soliton solutions with integer values of the topological or baryon charge, and in low energy limit it must reproduce the prototype Lagrangian (Zahed and Brown, 1986; Nikolaev, 1989)

$$\mathcal{L}_{pr} = -\frac{f_\pi^2}{4}\mathrm{Tr}L_\mu L^\mu. \tag{15}$$

We thus consider the Lagrangian

$$\mathcal{L} = f_\pi^2 \mathrm{Tr}\left(-\frac{1}{4}L_\mu L^\mu + \lambda\left(L_\mu L^\mu\right)^2\right), \tag{16}$$

where λ is the scale parameter of our model, and L_μ was defined in the introduction.

As usual, we use the spherically symmetric ansatz

$$U = e^{iF(r)\mathbf{n}.\boldsymbol{\tau}} \quad , \quad \mathbf{n} = \frac{\mathbf{r}}{|\mathbf{r}|}. \tag{17}$$

The energy of such a field is the functional

$$E^\lambda[F] = -8\pi f_\pi^2 \int_0^\infty \Lambda\left(\lambda\Lambda - \frac{1}{4}\right) r^2 dr, \tag{18}$$

where

$$\Lambda = F'^2 + \frac{2\sin^2 F}{r^2}. \tag{19}$$

Using the minimum energy variational principle, we obtain the following differential equation for the amplitude $F(r)$

$$\left(r^2\left(12\lambda F'^2 - \frac{1}{2}\right) + 8\lambda\sin^2 F\right)F'' + 8\lambda\left(2rF'^3 - \frac{\sin^2 F\sin 2F}{r^2}\right) - rF' + \frac{\sin 2F}{2} = 0. \tag{20}$$

This nonlinear differential equation is a boundary value problem and can be solved by numerical methods. By imposing proper boundary conditions $F(0) = N\pi$ where $N \in \mathbb{Z}$ and $F(r \to \infty) = 0$, one can find numerical solutions. First, we use the standard procedure to find asymptotic behavior near zero, ($r = 0$, $F(0) = N\pi$)

$$F(r) = N\pi + ar - \frac{2(16\,\lambda\, a^2 - 1)}{15(40\,\lambda\, a^2 - 1)}a^3 r^3 + \underline{O}(r^5), \tag{21}$$

where $F'(0) = a$.

Solutions with asymptotic (21) at origin are the Chiral Soliton solutions of our model which are proposed to describe baryons. Their topological charge is defined by equation (2) as usual. The solitons with baryon number $B = 1, 2$ and 3 are presented in Fig. 1. The scale parameter λ is fixed such that the solution with $B = 1$ corresponds to a nucleon.

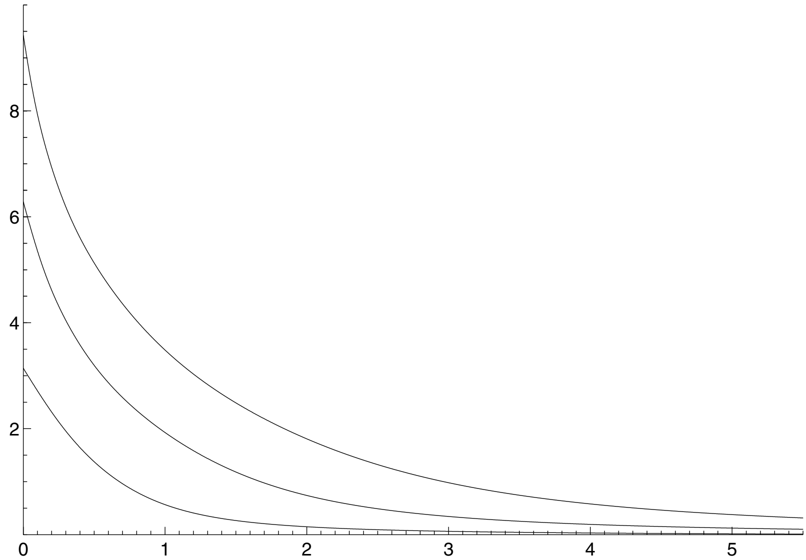

Fig. 1. Solitons with $B = 1, 2$ and 3. Horizontal axis: r (in fm). Vertical axis: $F(r)$.

The stability of the chiral soliton solutions can be easily checked by using the procedure employed in proving Derrick's theorem (Derrick, 1964). We rewrite the energy integral as

$$\mathcal{E} = 8\pi f_\pi^2(\mathcal{I}_1 + \mathcal{I}_2), \tag{22}$$

where

$$\mathcal{I}_1 = \frac{1}{4}\int_0^\infty \left(F'^2 + \frac{2\sin^2 F}{r^2}\right) r^2 dr, \tag{23}$$

$$\mathcal{I}_2 = -\lambda \int_0^\infty \left(F'^2 + \frac{2\sin^2 F}{r^2}\right)^2 r^2 dr \tag{24}$$

and $F(r)$ is a localized solution of $\delta\mathcal{E} = 0$. A necessary condition for a solution to be stable is that the second-order variation $\delta^2\mathcal{E} \geq 0$. Define $F_\kappa(r) = F(\kappa r)$ where κ is an arbitrary constant, and rewrite the energy integral for F_κ to obtain

$$\begin{aligned}\mathcal{E}_\kappa &= 8\pi f_\pi^2 \int_0^\infty \left[\frac{1}{4}\left(F_\kappa'^2 + \frac{2\sin^2 F_\kappa}{r^2}\right) - \lambda\left(F_\kappa'^2 + \frac{2\sin^2 F_\kappa}{r^2}\right)^2\right] r^2 dr \\ &= \frac{\mathcal{I}_1}{\kappa} + \kappa\mathcal{I}_2 \end{aligned} \tag{25}$$

Let us change the variable of integration from r, to λr; whence

$$\left(\frac{d\mathcal{E}_\kappa}{d\kappa}\right)_{\kappa=1} = -\mathcal{I}_1 + \mathcal{I}_2, \tag{26}$$

$$\left(\frac{d^2\mathcal{E}_\kappa}{d\kappa^2}\right)_{\kappa=1} = 2\mathcal{I}_1. \tag{27}$$

Since F_κ is a solution of $\delta\mathcal{E} = 0$ for $\kappa = 1$, we must have

$$\left(\frac{d\mathcal{E}_\kappa}{d\kappa}\right)_{\kappa=1} = 0, \qquad \mathcal{I}_1 = \mathcal{I}_2$$

$$\left(\frac{d^2\mathcal{E}_\kappa}{d\kappa^2}\right)_{\kappa=1} = 2\mathcal{I}_1 = \mathcal{I}_2 > 0, \tag{28}$$

since $\mathcal{I}_2$ is certainly a positive quantity. That is, $\delta^2\mathcal{E} > 0$ for a variation corresponding to a uniform stretching of the particle. Hence the solution $F(r)$ is stable.

Before closing this section, let us consider the question of the behavior of the spherically symmetric solutions with B. In the Skyrme model, $\mathcal{E}(B) \sim B^2$ (Bogomolny and Fateev, 1983). In the present model, the situation is different. The values of the total energy for different solitons $B = 1, \ldots, 10$ are presented in Table I. By interpolating these data it can be shown that the expression for $\mathcal{E}$ as a function of B would have the form

$$\mathcal{E}(B, \lambda = -1) \simeq 1.0731B^3 + 4.3656B^2 + 4.554B + 0.159. \tag{29}$$

The graph corresponding to this fit is plotted in Fig. 2, together with the data from Table I.

Table I. The Energy $\mathcal{E}(B, \lambda = -1)$ of Different Solitons (Baryons)

Baryon charge, B	Energy of solitons, $E(\lambda = -1)$
1	10.193
2	35.280
3	182.03
4	156.89
5	266.27
6	416.56
7	614.07
8	865.38
9	1177.00
10	1555.40

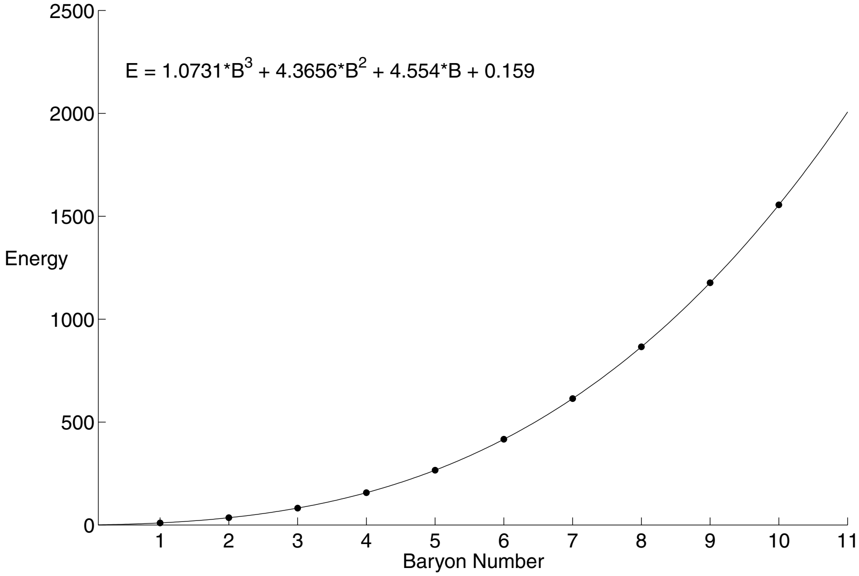

Fig. 2. The energy $\mathcal{E}$ of different solitons as a function of baryon number B.

4. CONCLUSION

In this article, we introduced a nonlinear electrodynamics which admits a finite value for the self energy of a point-like charge. A chiral model (which corresponds to an effective chiral theory) was built with a similar form for the Lagrangian. The resulting chiral solitons have finite energies. These solutions were treated as baryon states.

The chiral solitons of our model were shown to be stable against radial deformations by applying a scale transformation (Derrick, 1964). In models like that of Pavlovsky's (2002), the solitons' stability is based on the stability of the prototype Lagrangian (15) which has well-known stable static topological solitons in 1+1 dimention (Polyakov and Belavin, 1975a,b).

In the present paper, we did not survey all aspects of the proposed model. Non-spherical solutions and the corresponding properties of such solutions and baryon interactions remain open questions. All of these questions should be the themes for a future investigation.

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